Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it means something entirely different

Johann Wolfgang von Goethe

Is Nature Algebraic or Geometric?

- Introduction
- Standard Newtonian Physics
- Geometrising Newtonian Physics
- Geometrising General Relativity
- Geometrising Differential Geometry
- Algebraicising Differential Geometry
- Conclusions

Introduction

- Algebra: theory of addition and multiplication of numbers. More generally: theory of operations such as multiplication and addition on mathematical objects such as numbers, matrices, and functions.
- Geometry: theory of Euclidean 3-dimensional space. More generally: theory of structure of all possible mathematical spaces, such as topological, metric, and differential structure of curved multidimensional spaces and fibre bundles.

A very brief history

- The Greeks regarded algebra and geometry as quite distinct subjects.
- Algebra: the mathematics of the discrete.
- Geometry: the mathematics of the continuous.
- Fermat and Descartes (early17th Century) unified the two into 'analytic geometry': functions correspond to regions in a space.
- Newton and Leibniz (late 17th Century) invented calculus: an algebraic way of solving geometrical problems, which was the royal road to modern science.

The unreasonable effectiveness of mathematics in the natural sciences

The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.

Eugene Wigner

My explanation

- Mathematics studies sets of simple rules (axioms) which give rise to complex and interesting structures
- Nature has an interesting and complex structure
- Scientists and Philosophers tend to think that the underlying rules (laws) which give rise to this interesting and complex structure are simple.
- So it is not surprising that the structure of the world is the same as (is isomorphic to) that of some (not all!) mathematical systems

The sense in which nature might be geometric or algebraic

- The previous explanation allows one to maintain mathematical nominalism: mathematical objects do not exist, only physical ones do, but the physical objects instantiate a structure that is isomorphic to some of the structures that mathematicians study.
- But which structures? Algebraic or geometric ones (or both, or other ones….)?

What some famous old people had to say on this matter

Geometry is the only science that it hath pleased God to bestow on mankind

Thomas Hobbes

Philosophy is written in that great book which ever lies before our eyes — I mean the universe — but we cannot understand it if we do not first learn the language and grasp the symbols, in which it is written. This book is written in the language of mathematics, and the symbols are triangles, circles and other geometrical figures, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth.

Galileo Galilei

Men of recent times, eager to add to the discoveries of the ancients, have united specious arithmetic with geometry. Benefitting from that, progress has been broad and far-reaching if your eye is on the profuseness of an output, but the advance is less of a blessing if you look at the complexity of its conclusions. For these computations, progressing by means of arithmetical operations alone, very often express in an intolerably roundabout way quantities which in geometry are designated by the drawing of a single line.

Isaac Newton

Standard Newtonian Physics

- Space is 3-dimensional and Euclidean. I.e. there exist ‘Cartesian’ coordinates x,y,z such that the distance between a location P with coordinates x_p,y_p,z_p and a location Q with coordinates x_q,y_q,z_q equals

  $$\sqrt{(x_p-x_q)^2+(y_p-y_q)^2+(z_p-z_q)^2}.$$

- Time is one-dimensional and Euclidean. I.e. there exist a ‘Cartesian’ coordinate t such that the size of the time lapse between events E and F equals t_f-t_e.

Standard Newtonian Physics: The Contents of Space and Time

- There is a gravitational field function $\phi(x,y,z,t)$ and a mass-density function $\rho(x,y,z,t)$. W.r.t. Cartesian coordinates x,y,z they satisfy:

$$\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2+\partial^2\phi/\partial z^2 = -k\rho.$$

- The Cartesian coordinates x(t), y(t), z(t) of a particle's trajectory satisfy:

$$\partial^2 x/\partial t^2=-c\partial\phi/\partial x,\ \partial^2 y/\partial t^2=-c\partial\phi/\partial y,\ \partial^2 z/\partial t^2=-c\partial\phi/\partial z$$

Standard Newtonian Physics is rife with algebraic and set theoretic objects

- A coordinate x(p) is a function from locations in space to the real numbers
- The gravitational field $\phi(x,y,z,t)$ is function from quadruples of real numbers to real numbers
- Differentiation is a map from functions to functions

Why it is problematic to accept sets as existing

- As soon as one accepts sets as existing one is committed to the existence of a huge amount of crazy objects: the empty set Ø, the set consisting of the empty set {Ø}. Also: {{{Ø}}}, {{2},{{Ø}}}, …..
- As soon as one accepts sets as existing there are all sorts of facts which are true or false but for which we can have no good reason to believe or disbelieve them: claims about the existence of infinitely large cardinals, the continuum hypothesis, ……

It is prima facie objectionable for numbers to occur in this way in physics

- It seems strange to think that how heavy a physical object is consists of some relation between that physical object and a number, i.e. an abstract object, and that how the physical object physically behaves depends on the relation that it stands it to such abstract objects

- Moreover It seems hard to explain why aspects of these numbers are conventional: different scales (pounds/kilograms/ounces) correspond to different numerical values of masses. So mass values are really relations between physical objects, SCALES and numbers? What are scales? Abstract objects? Concrete objects? As soon as you start to think of scales as concrete objects that function as conventionally chosen standards you are already halfway to the geometrical way of thinking about mass values which I will now present.

The purely geometric features of time

- Time-Between(I_1,I_2,I_3): Instant I_2 is between instant I_1 and instant I_3.
- Time-Congruent(I_1,I_2, I_3, I_4): The temporal interval between instants I_1 and I_2 is just as long as the temporal interval between $I_3$and I_4
- If one lays down a few simple axioms, one can show that one can represent all the congruence and betweenness facts by a function t(I), from instants I to real numbers, i.e. by a coordinate representation of time,
- That function, that coordinatisation of time, will be unique up to linear transformations if one demands that the difference in coordinates is equal when the intervals are congruent.

The geometric features of space

- Assume spatial betweenness and spatial congruence relations subject to some simple axioms. One can again prove representability of these betweenness and congruence facts by triples of real numbers, i.e. coordinates.

- If one demands that the coordinate function

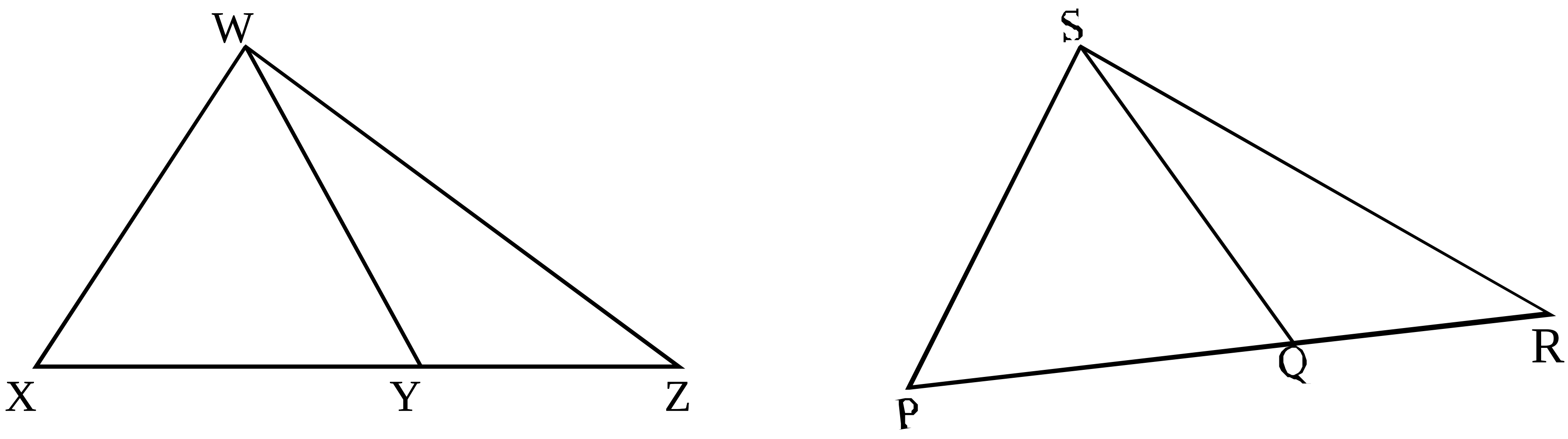
$$\sqrt{(x_p-x_q)^2+(y_p-y_q)^2+(z_p-z_q)^2}$$

  takes equal values when straight line segments are congruent then this coordinate representation is unique up to linear transformations. These are the Cartesian coordinates.

Example of an axiom

If $x \neq y$ & Bet(xyz) & Bet(pqr) & Cong(xypq) &
Cong(yzqr) & Cong(xwps) & Cong(ywqs)

Then Cong(zwrs)



A geometrical representation of the masses of particles

- Assume the existence of a ‘Mass Space’
- Assume that particles occupy locations in mass space.
- Assume a Mass Betweenness relation and Mass Addition relation between points in Mass Space, subject to certain axioms which give Mass Space a geometrical structure
- These relations again have a numerical representation as ‘Mass values’ which is unique up to linear re-scaling if one demands that differences in Mass values are identical when the Mass Space intervals are congruent.
- This explains

  a) why Mass properties can be represented by real numbers

  b) why this representation is conventional up to linear re-scalings , i.e. why you can represent masses as numbers of pounds, or kilograms, or …..

Same for gravitational fields

- Assume there is a gravitational field strength space, and field strength betweenness relations and fields strength addition relations subject to certain axioms
- Assume an occupation relations between the gravitational field and ordinary space and time and gravitational field strength space

Completing the job

- One can prove that one can then write simple axioms in terms of betweenness, congruence, mass betweenness, mass congruence, gravitational field-strength betweenness and gravitational fields strength congruence which are equivalent to standard Newtonian gravitational theory.

General Relativity

- Space-time is curved
- Space-time does not split into space and time
- The fundamental notion of distance is that of length of path
- Distances can be positive as well as negative.
- Between any two points there exists a path of arbitrarily large negative length; there is no shortest path between points.

A problem for geometrisation

- In General Relativistic space-time there are still things analogous to straight lines, namely ‘geodesics’
- A geodesic is a path in space-time such that any small variation of that path makes it longer (or shorter).
- But, there can be multiple geodesics between the same pair or points. So one can not say: consider ‘the’ geodesic between points p and q.
- So the notion of p lying between q and r (on ‘the’ geodesic that runs from q to r) does not make straightforward sense in General Relativity.

A geodesic-based geometrisation strategy

- Useful fact: any general relativistic space-time can be divided into overlapping patches P such that for any pair of points in a single patch P there is a unique geodesic between those two points.
- Assume patch-dependent space-time betweenness and space-time congruence relations.
- One can then prove a representation theorem: if a set of patch dependent space-time betweenness and congruence relations has a representation as a general relativistic space-time, then that representation is unique up to global linear multiplications of the metric tensor.

The contents of space-time

- Scalar fields on space-time: scalar field value betweenness and addition relations between space-time points
- Vectors and vector fields. A vector at point p can be represented by a pair of points p,q. Think of it as an arrow stretching from p to q along the geodesic between p and q. It's length is just the length of the geodesic between p and q, and its direction is the direction in which that geodesic 'points' at p. A vector field consists of a vector at each point p in space-time.
- Caution: there is no path independent notion of vectors at different points pointing in the same direction.

Completing the job

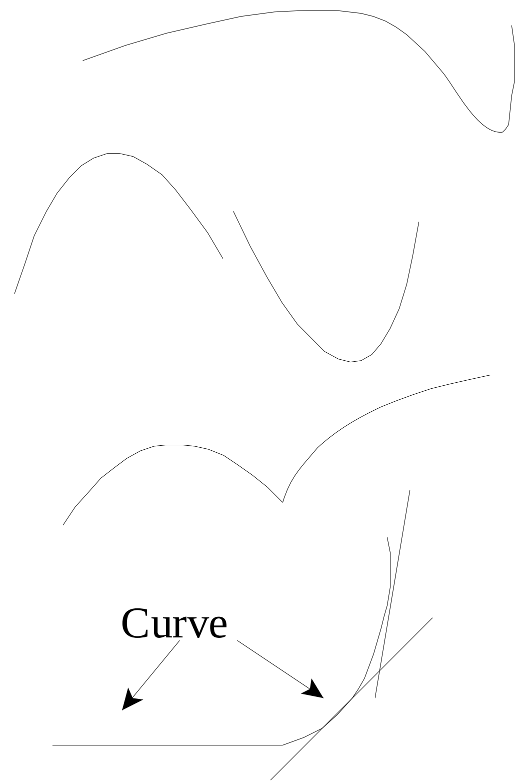
- Since we have a representation theorem, we can in principle state the laws of General Relativity in purely geometric terms. But we have no idea as to how to state these laws.
- Even worse: there are heuristic arguments that such an axiomatisation must be horribly, horribly complicated.

More reasons to dislike the geodesic approach to geometrisation of GR

- In modern physics one starts off with a differentiable manifold, a space that has topological and differential structure. Metric structure is then introduced via a local metric tensor field. Geodesics can then defined. While metric structure can in principle be recovered from geodesic structure, the metric tensor field seems natural and fundamental, geodesic structure seems derivative.
- Many theories of modern physics, gauge theories in particular, is set in spaces that do not have metric structure. We want to be able to geometrise such theories too. In order to do this we need to be able to geometrise differentiable manifolds.

What's a differentiable manifold?

- It is a space with topological and differential structure
- Topological structure: notion of continuous lines, connected regions. This can be nominalistically defined by specifying which regions are 'open', subject to certain axioms
- Differential structure: distinguishes curves with 'kinks' from curves without a 'kink'. And it distinguishes once differentiable from twice differentiable curves. And so on.



Twice differentiable: angles of tangents change in a 'differentiable' manner.

Standard way to put differential structure on a manifold

- Provide the manifold M with an ‘Atlas’, i.e. divide M up into patches P, provide each patch with n coordinates, i.e. provide a 1-1 mapping from each patch P of M to a patch of R^n. Or, less crazily specific: specify an equivalence class of coordinate systems, each of which is smooth w.r.t. each other.
- Then say that a curve in M, i.e. a map from R to a line in M, is k times differentiable iff the map from R from R^n induced by an allowed coordinate system is k times differentiable.
- Note that all of this is rife with algebraic and set theoretic objects.

Three strategies for geometrising differential geometry (work by Cian Dorr and me)

- Make do with only regions of space-time
- Assume additionally the existence of a scalar field value space
- Assume additionally the existence of a vector bundle space (which could be the tangent bundle space)

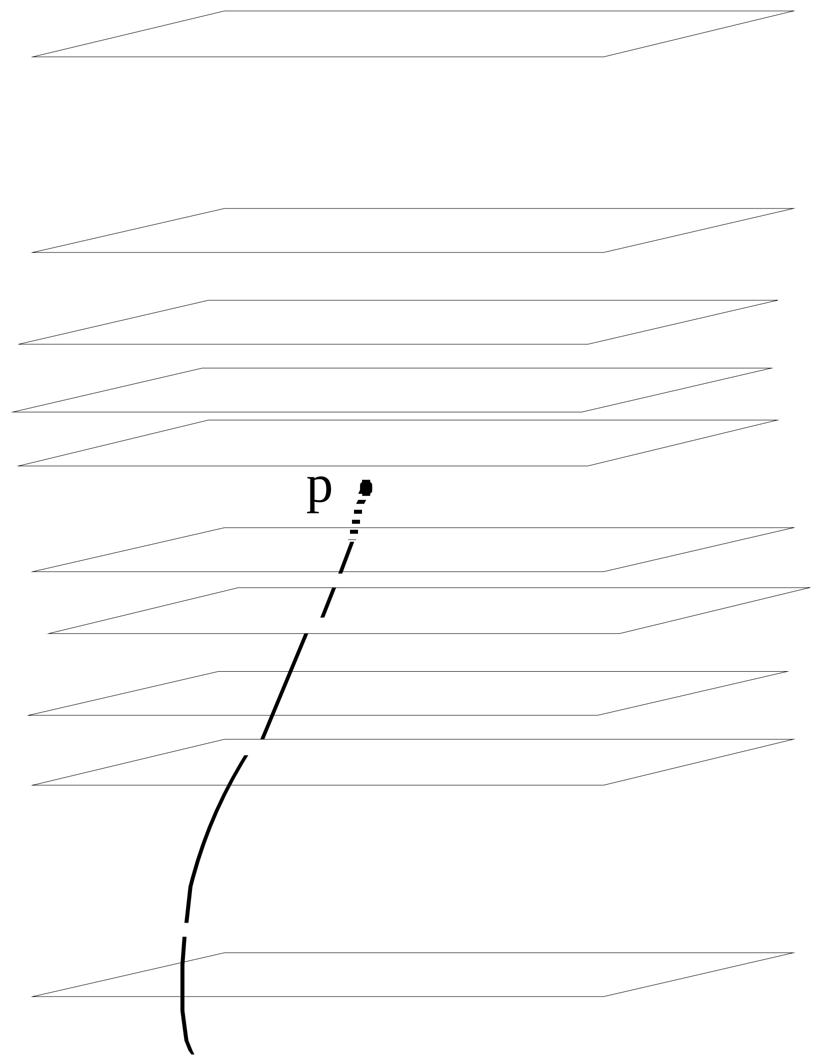
Making do with regions in space

- First idea: to give a space differential structure we just specify which lines, surfaces, etc in the space are smooth. I.e. we assume a primitive property (predicate) ‘Smoothness’ which applies to regions. We lay down some axioms, and we say that any assignment of smoothness properties to regions that satisfies these axioms is a differentiable structure for that space.
- Problem: one can prove that different differential structures can have the same classification of regions into smooth and non-smooth

Next attempt to make do with regions

- Have a physical representation in space of vectors and co-vectors (1-forms). Assume that vectors and co-vectors have a fundamental property 'Kosherness', where intuitively a 'Kosher' vector is one that is tangent to a smooth line, and a non-kosher one is not. Then lay down axioms on 'Kosherness' so that any set of Kosher vectors subject to these axioms corresponds to a differentiable manifold

Vectors and co-vectors as regions



The bad news

- We couldn’t find an axiomatisation using the ‘Kosherness’ predicate alone which guarantees that the resulting structure is isomorphic to a differentiable manifold. With an additional predicate we could, but it was very unnatural and complicated

Second approach: scalar value space

- This approach is inspired by axiomatisations of differential geometry pioneered by Chevalley, Nomizu, Sikorski and Penrose and Rindler.
- One can specify the differential structure of a (topological) manifold by saying which scalar functions on the manifold are smooth, subject to certain axioms on smoothness.
- As it stands we cannot use this approach since it makes use of functions (set theoretic entities) from the manifold to the real numbers (algebraic entities)

Our strategy

- We assume the existence of a physical space, scalar value space. This space has a certain structure, which we give by laying down certain simple axioms on regions in this space in terms of the primitive predicates 'Addition', 'Positivity' and 'Unit'. One can think of this space as consisting of points that can be occupied (or not occupied) by scalar fields
- We then define 'scalar functions' as certain regions in this space.
- Finally we lay down an analogy of Penrose and Rindler's axiom on the 'Smoothness' of such regions.

Pros and cons

- The axiomatisation is simple and equivalent to a standard axiomatisation of differentiable manifolds
- We in effect imposed on scalar value space the same algebraic structure as the real numbers have: Addition, Positivity and Unit allow one to define the notion of ‘multiplying’ points in scalar value space.
- But we do not have set theoretic structure: we have no need for sets, sets of sets, …. We also have no need for the mathematical notion of functions and maps as sets (ordered pairs). The only sense of functions and maps that we need are regions in physical scalar value space.

Third strategy: assume the existence of a vector bundle space (which could be the tangent bundle)

- We assume the existence of a physical vector bundle space. We do not need the existence of scalar value space. A physical vector field is then a region in this physical vector bundle space.
- This vector bundle space just has a vector space structure (without inner product). We do not need a structure analogous to multiplication in order to characterise a differentiable manifold.
- The axiomatisation we give is, arguably, relatively simple

Geometrising other aspects of modern physics

- It is easy to extend the above geometrical axiomatisations to general relativity and gauge theories, since these just need the basic machinery of differential geometry
- How about quantum mechanics?
- That depends on your favourite formulation of quantum mechanics.

For instance

- If you are a realist about wavefunctions, we can geometrise it by specifying the geometrical structure of the (complex) space that wave-functions occupy.
- If you are a realist about Hilbert spaces, we can geometrise this by giving the geometrical structure of Hilbert spaces (in terms of a vector space and inner product structure).

Algebraicising differential geometry

- Again inspired by Penrose and Rindler style axiomatisations of differential geometry we can also take an algebraic approach.
- We assume the existence of a 'scalar function space'
- Intuitively each complete scalar function is a **point** in this space.
- There two primitive relations between such points: addition and multiplication.
- These relations are algebraic relations. (Intuitively between entire scalar functions over the manifold, but on our approach they hold between the **points in our gigantic function space**.)

Good news and bad news

- One can show that the set of all addition and multiplication facts between entire scalar functions uniquely determine the differential structure of a manifold on which these functions are imagined to be the smooth scalar functions
- We do not need any set theory.
- However, it is totally unclear how to characterise axiomatically in terms of these addition and multiplication relations the structure of particular differentiable manifolds that we are interested in, such as R^4 with its standard differential structure.

A false dichotomy?

- The fact that we might be able to exploit Penrose and Rindler style axiomatisations of differential geometry to give both a geometrical axiomatisation and an algebraic axiomatisation suggests that perhaps these are two sides of the same coin: neither is more correct than the other. The question whether the structure of the world is algebraic or geometric is a false dichotomy.

Duality of the algebraic/geometric division?

- Often one can characterise a geometry algebraically by giving the algebra of the transformations that leave the (local; global is harder) geometric structure invariant. Similarly one can characterise certain algebras by specifying certain a space with a geometrical structure and then finding the algebra of the transformations which leave that (local or global) geometrical structure invariant.

An example of duality

- One can do quantum field theory by giving the structure of a Fock-space (Hilbert space), but one also do it by specifying the structure of the algebra of observables
- These two are connected by the ‘GNS-representation theorem’. (Note that some algebras of observables do not have a Hilbert-space representation.)
- It is not clear (to me at least) which allows for a simpler axiomatisation

Tentative conclusion

- Geometrising is a pervasive feature of modern physics, especially gauge theories. With hindsight we can even think of ordinary physical space as just another case of geometrising, namely the geometrising of the position properties of objects.
- We can geometrise differential geometry, and many other theories of modern physics, despite the fact that standard formulations make extensive use of real numbers, functions, functions of functions and so on up the set theoretic hierarchy.

But

- We might also be able to algebraicise differential geometry and other modern theories of physics
- If these axiomatisations turn out to be equally simple and natural then perhaps the dichotomy between algebra and geometry is a false dichotomy after all….